Title: The Chain Rule
Class: Math 130 or Math 150
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Instructions to tutor: Read instructions and follow all steps for each problem exactly as given.
Keywords/Tags: Calculus, derivative, composition, chain rule

The Chain Rule

Purpose:
This is intended to strengthen your ability to find derivatives using the chain rule.

To this point, we have seen the power rule, the product rule, and the quotient rule. These are three of the 'big four' derivative rules. Now we need to look at compositions of functions, such as $y=(x^2+3x+5)^4$, $y=e^{3x^3-5x+4}$, and $y=\sqrt{x^2-4x+7}$. Note that each of these can be written in the form $y=f(g(x))$, where there is an *inner* function and an *outer* function.

First we look at the derivative of a function to a power, known as the *Generalized Power Rule.*

Generalized Power Rule: $\frac{d}{dx}\left[g(x)\right]^n = n\left[g(x)\right]^{n-1}\cdot g'(x)$

Notice how this looks a lot like the *Power Rule,* $\frac{d}{dx}x^n = nx^{n-1}$. The difference now is that instead of just the independent variable *x*, we have a function $g(x)$. Let's look at a couple examples.

Example: Find $\frac{d}{dx}(x^2+3x+5)^4$.

Note that this is of the form $\frac{d}{dx}\left[g(x)\right]^4$, where $g(x)=x^2+3x+5$.

The Generalized Power Rule says to take the derivative of the outer function, in this case $(\textit{something})^4$, and multiply by the derivative of *something.*

So, this translates to $\frac{d}{dx}\left[\qquad\qquad\right]^4 = 4\left[\qquad\qquad\right]^3\cdot\frac{d}{dx}\left[\qquad\qquad\right]$.

Filling in the blanks, $\frac{d}{dx}\left[x^2+3x+5\right]^4 = 4\left[x^2+3x+5\right]^3\cdot\frac{d}{dx}\left[x^2+3x+5\right]$.

This gives $\frac{d}{dx}\left[x^2+3x+5\right]^4 = 4\left[x^2+3x+5\right]^3\cdot\frac{d}{dx}\left[x^2+3x+5\right] = 4\left[x^2+3x+5\right]^3(2x+3)$.

Let's look at another example, this time involving a square root as the outer function.

Example: Find $\dfrac{d}{dx}\sqrt{3x^5 - 5x^2 + 2x + 9}$.

Note again that this is of the form $\dfrac{d}{dx}\Big[g(x)\Big]^{1/2}$, where $g(x) = 3x^5 - 5x^2 + 2x + 9$.

Fill in the blanks: $\dfrac{d}{dx}\Big[\qquad\qquad\qquad\qquad\Big]^{1/2} = \dfrac{1}{2}\Big[\qquad\qquad\qquad\qquad\Big]^{-1/2} \cdot \dfrac{d}{dx}\Big[\qquad\qquad\qquad\qquad\Big]$.

Now finish this:

Did you obtain $\dfrac{d}{dx}\sqrt{3x^5 - 5x^2 + 2x + 9} = \dfrac{1}{2}\Big(3x^5 - 5x^2 + 2x + 9\Big)^{-1/2}(15x^4 - 10x + 2)$?

Note this may be written as $\dfrac{d}{dx}\sqrt{3x^5 - 5x^2 + 2x + 9} = \dfrac{15x^4 - 10x + 2}{2\sqrt{3x^5 - 5x^2 + 2x + 9}}$.

Try the next couple on your own.

1. Differentiate each function.

(a) $f(x) = (x^3 - 4x^2 + 6x - 13)^5$

(b) $f(x) = \sqrt[3]{5x - x^2}$

Check your answers:

1. (a) $f'(x) = 5(x^3 - 4x^2 + 6x - 13)^4(3x^2 - 8x + 6)$ (b) $f'(x) = \dfrac{5 - 2x}{3\sqrt[3]{(5x - x^2)^2}}$

The Chain Rule: $\dfrac{d}{dx} f\left[g(x)\right] = f'(g(x)) \cdot g'(x)$

The general statement of the chain rule, for a composition of functions, says to take the derivative of the outer function, evaluated at the inner function, times the derivative of the inner function.

Let's apply this general formula to a natural logarithm function.

Example: Find $\dfrac{d}{dx}\ln(x^2 - 5x + 3)$.

Note that the inner function is $g(x) = x^2 - 5x - 3$ and the outer function is $f(x) = \ln x$.

Now, $f'(x) = \dfrac{1}{x}$ and $g'(x) = 2x - 5$.

So $\dfrac{d}{dx}\ln(x^2 - 5x + 3) = \dfrac{1}{x^2 - 5x + 3} \cdot (2x - 5) = \dfrac{2x - 5}{x^2 - 5x + 3}$.

You can probably see that a formula for the chain rule with natural logarithms is given by $\dfrac{d}{dx}\ln(f(x)) = \dfrac{f'(x)}{f(x)}$, though there is no need to memorize this formula if you remember the general form of the chain rule. Try the next one on your own.

2. Find $\dfrac{d}{dx}\ln(4 + 3x - x^2 + 7x^4)$.

Did you obtain $\dfrac{d}{dx}\ln(4 + 3x - x^2 + 7x^4) = \dfrac{3 - 2x + 28x^3}{4 + 3x - x^2 + 7x^4}$? Good! Now let's move on to the natural exponential function.

Example: Find $\dfrac{d}{dx}e^{4-x^2}$.

Note that this is of the form $\dfrac{d}{dx}e^{g(x)}$, where $g(x)$ is the inner function and $f(x) = e^x$ is the outer function.

So $g'(x) = -2x$ and $f'(x) = e^x$.

By the Chain Rule, $\dfrac{d}{dx}e^{4-x^2} = e^{4-x^2} \cdot (-2x^2)$.

Again note that this is the derivative of the outer function, evaluated at the inner function, times the derivative of the inner function.

Try the next one on your own.

3. Find $\dfrac{d}{dx}e^{4x^3-3x^2+3x-7}$.

Did you obtain $\dfrac{d}{dx}e^{4x^3-3x^2+3x-7} = (12x^2 - 6x + 3)e^{4x^3-3x^2+3x-7}$? Good! Try the remaining problems to check that you have mastered the Chain Rule.

4. Find the derivative of each of the following using the Chain Rule.

(a) $f(x) = \left(4x^3 + 2x^2 + 5\right)^{12}$

(b) $f(x) = \dfrac{1}{\sqrt{x^2 - 4x}}$

(c) $f(x) = x\ln(x^2 + 5x + 1)$

(You should first apply the product rule, and then you will see a chain rule.)

(d) $f(x) = \dfrac{x}{e^{x^2+2x}}$

(You should first apply a quotient rule, and then you will see a chain rule.)

Check your answers:

4. (a) $f'(x) = 12(12x^2 + 4x)(4x^3 + 2x^2 + 5)^{11}$ (b) $f'(x) = -\dfrac{2x-4}{2\sqrt{(x^2-4x)^3}}$

(c) $f'(x) = \ln(x^2 + 5x + 1) + x \cdot \dfrac{2x+5}{x^2+5x+1}$ (d) $f'(x) = \dfrac{e^{x^2+2x} - x(2x+2)e^{x^2+2x}}{\left(e^{x^2+2x}\right)^2}$

(These solutions may be further simplified, but they should let you know if you are on the right track.)